Activity Sheet 1: David Blackwell and the Theory of Duels

David Blackwell is cited as one of the pioneers in the theory of duels. He describes how he and his colleagues became interested in duels while working for the Rand Corporation in the late 1940's:

> One day some of us were talking and this question arose: If two people were advancing on each other and each one has a gun with one bullet, when should you shoot? If you miss, you're required to continue advancing. That's what gives it dramatic interest. If you fire too early your accuracy is less and there's a greater chance of missing. **It** took us about a day to develop the theory of that duel... Then I got the idea of making each gun silent. With the guns silent, if you fire, the other fellow doesn't know, unless he's been hit. He doesn't know whether you fired and missed or whether you still have the bullet. That turned out to be a very interesting problem mathematically.
>
> (Albers & Alexanderson, 1985, p. 25)

His research, which he completed with various coauthors, was among the first rigorous analysis of the age-old concept of a duel, a concept that had many applications in the Cold War era. In this activity sheet, we will explore a simplified version of his work.

In the classic pistol duel, the two duelists start back to back. On command they march a prescribed number of paces away from each other, then turn to face each other, pistols at their sides. Then they raise their smooth bore pistols and each fire a single bullet at the other. If one duelist hits the other, he is considered the winner, while if both are hit or both are missed the duel is considered a draw. It seems to make sense that firing your pistol quickly might be advantageous as you could hit your combatant before he fires and hence throw off his shot. On the other hand, if you wait longer (and presumably aim more accurately) you stand a better chance of hitting your opponent. If you wait long enough to see that your opponent fired his weapon, and you are still standing, you could take as long as you wanted to line up your shot.

We will simulate two simplified versions of this dual. The first will be a "noisy duel" where each duelist can tell when his opponent fires his weapon, and the second a "silent duel" where this information is not available. For both of these games we will make the following assumptions:

- The duelists must fire their weapons after either 1,2,3,4,5, or 6 seconds.
- If a duelist fires after second i, the probability of hitting their opponent is $i/6$.
- In the noisy duel, if the duelists do not fire their guns at the same time, the duelist who fires second will wait until after 6 seconds thus guaranteeing that he hits his opponent.

Game 1: The Noisy Duel

To simulate this game we need two duelists with a piece of paper and a pair of dice. Before the game starts, each duelist writes down a number from 1 to 6 representing when he plans to fire his weapon. The two numbers are then compared. If the numbers are not equal, the player with the lower number will roll his die to simulate firing his weapon. If his number is i and he rolls 1,2,... ,i then he hits his target and wins the duel. If he misses, with a roll of i+ 1,. . .6, then the other player will win as he will wait until after 6 seconds to fire and will be guaranteed a hit. If both players picked the same number, they both roll their die to simulate firing their weapon and determine the outcome in the same manner.

Activity 1: Below are some strategies represented as ordered pairs. The pair *(i,j)* represents Player 1 planning to fire after i seconds and Player 2 planning to fire after j seconds. Below each ordered pair is one or two die rolls. In each case, determine the outcome of the duel.

Strategy	(2,4)	(5,1)	(4,4)	(1,6)	(3,5)	(1,6)
Roll	3	I	3, I	5	3	2
Winner						

Activity 2: Now get some dice and find a partner to challenge to a duel. Simulate 10 or 15 duels, and try to find a winning strategy. Hypothesize what you think the optimal strategy might be.

Game 2: The Silent Duel

This game is simulated in the same manner as the noisy duel unless the player who fires first misses. In this case, the other player does not know he has fired and will continue to fire his weapon as planned. Hence two rolls are always needed, and it becomes more likely that both players will miss.

Activity 3: Analyze the strategies and rolls given below and see if you can determine the outcome of each duel. The roll on the left is always Player l's roll and the roll on the right is Player 2's roll, regardless of the order in which they fired.

Strategy	(2,4)	(5,1)	(4,4)	(1,6)	(3,5)	(1,6)
Roll	3,3	1,3	3,1	5,2	3,6	2,3
Winner						

Activity 4: Get your dice and partner and prepare to duel. Try to hypothesize an optimal strategy as you simulate 10 or 15 duels.

Mathematical Analysis of the Duel

To analyze these simplified duels we must use some probability theory. In particular, we must look at the expected value of each strategy. To do this we assign a numerical value to each possible outcome of the game. For simplicity we will assign a weight of 1 to a duel in which Player 1 wins and a weight of -1 to a duel in which he loses. A draw will be assigned a value of O. Now to determine the expected value of each strategy we simply take the sum of each possible outcome times its probability of occurring. One way to think about the expected value of a strategy is that it is the average of the outcomes if this strategy is employed a large number of times. In a duel this may be of limited use since losing once can be quite disastrous.

Since our outcomes are based on a roll of the dice, these probabilities are easy to compute. In real life the calculation may be far more difficult. Also notice that we don't need to compute the probability of a draw in each strategy as this will be multiplied by the value of a draw: O. This will greatly simplify our computations.

Game 1: The Noisy Duel

Let's look at the expected value of the strategy (3,2) in a noisy duel. If this strategy is employed, Player 2 will fire his weapon after 2 seconds. If he hits Player 1, which happens with probability 2/6, Player 1 loses and the duel has a value of -1. If Player 2 misses, which happens with probability 4/6, Player I waits for 6 seconds and hits Player 2 with probability 1. In this case the game has a value of 1. Hence the expected

value of this strategy is: $(-1)X\,2/6+(1)X\,4/6=2/6$

Activity 5: Compute the expected values of the following strategies; (2,5), (5,1), (2,2), (3,3), (3,4), and (3,2) in the noisy duel. The entry in the *i*th row and jth column of the table given below is the expected outcome of strategy *(i,j).*

j / i	1	2	3	4	5	6
1	0	-4/6	-4/6	-4/6	-4/6	-4/6
2	4/6	0	-2/6	-2/6	-2/6	-2/6
3	4/6	2/6	0	0	0	0
4	4/6	2/6	0	0	2/6	2/6
5	4/6	2/6	0	-2/6	0	4/6
6	4/6	2/6	0	-2/6	-4/6	0

If we are Player 1 and we wish to live through our duel, we want to choose as our strategy the row that has the largest minimum value. This strategy minimizes our chances of losing no matter what strategy Player 2 adapts. Obviously rows 3 or 4 maximize the minimum at 0, that is no matter what Player 2 does, if Player 1 picks strategy 3 or 4 he has at least as good a chance of winning as losing. One could even make the argument that strategy 4 is better than strategy 3 as our chances of winning against strategy 5 or 6 (should Player 2 be foolish enough to choose either of those strategies) would be greater than with strategy 3. This may seem counterintuitive as it says that in a duel where missing before you opponent fires guarantees losing, waiting until you have better than a 50% chance of success is a better strategy.

Game 2: The Silent Duel

The main difference in the analysis of the silent duel is that if the first person misses his shot, this does not guarantee his death. If we look at the expected outcome of strategy (3,2) again we see that Player 2 shoots first and hits Player 1 with probability 2/6. If he misses, which happens with probability 4/6, Player 1 still has to hit his shot, which happens with probability 3/6. Thus the expected value of this strategy is

$(-1)x2/6+(1)x4/6x3/6=0$ So we see immediately that the silent duel does indeed have different outcomes from the noisy duel.

Activity 6: Compute the expected values of the following strategies; (2,5), (5,1), (2,2), (3,3), (3,4), and (3,2) in the silent duel. The table below gives the expected value for all of the strategies for the silent duel. The entry in the *i*th row *j*th column gives the value for strategy *(i,j)*.

	1	2	3	4	5	6
1	0	-4/36	-9/36	-14/36	-19/36	-24/36
2	4/36	0	0	-4/36	-8/36	-12/36
3	9/36	0	0	6/36	3/36	0
4	14/36	4/36	-6/36	0	14/36	12/36
5	19/36	8/36	-3/36	-14/36	0	16/36
6	24/36	12/36	0	-12/36	-16/36	0

Activity 7: Determine the optimal strategy for each player in the silent duel. For Player 1 this is the row with the maximum minimum value and for Player 2 the column with the minimum maximum value.

Conclusion: There are many variations of this simple model of a duel. For example, we could let the probabilities of the two duelists hitting their shots increase at different rates, or we could assume that they had more bullets in their pistols. Of course in real life, time does not increase in a discrete fashion and so the duelists could fire at any time between 0 and 6. According to David Blackwell, "[two-person duels] are the games for which the theory is clear and beautiful" (Albers & Alexanderson, 1985, p. 25). We hope that you have enjoyed these games and that you have developed an appreciation for David Blackwell's work on the theory of duels.

Activity Sheet 2: **David Blackwell and The Prisoner's Dilemma**

David Blackwell, one of the greatest African American mathematicians, is interested in the theory of games. We'll look at one such game called the prisoner's dilemma.

Prisoner's Dilemma Scenario: Imagine that you and your accomplice have robbed a bank. Outside of the bank you are apprehended by police, separated, and then taken to different interrogation rooms in the police station. The police offer you a deal. You have to choose whether or not to implicate your accomplice. If both of you implicate each other then you and your accomplice will each go to prison for 2 years. However, if one of you implicates the other but the other keeps silent, the one who has ratted out his accomplice will go free, while the other will rot in jail for 5 years on the maximum charge. If you both keep silent, only circumstantial evidence exists, and so you will both serve one year.

Question 1. Fill in the following table to help organize the ramification of each option:

	Your Accomplice Implicates You	Your Accomplice Keeps Silent
You Implicate Your Accomplice	You receive: ____years Accomplice receives: ____years	You receive: ____years Accomplice receives: ____years
You Keep Silent	You receive: ____years jailtime Accomplice receives: ____years	You receive: ____years Accomplice receives: ____years

Question 2: If you can talk to your accomplice and you trust him or her, what should you do to minimize the time that you both spend in jail? Explain why. This is called the co-operative strategy.

Question 3: Let's say that you know that your accomplice is going to implicate you. What should you do to minimize your jail time? Compare your options and explain.

Question 4: Let's say that you know that your accomplice is going to keep silent. What should you do to minimize your jail time? Compare your options and explain.

Question 5: You should have gotten the same answer for Questions 3 and 4. Following this strategy is best for you if you can't trust your accomplice (who, after all, is a criminal) because you come out ahead no matter what the other person does. This is called the selfish strategy. If both you and your accomplice follow the selfish strategy, how much time will you each spend in jail?

David Blackwell explains that:
The situation with the Soviet Union has [had] elements like this in it. To cooperate is to disarm and to double-cross is to re-arm with bigger and bigger weapons. That takes a lot of resources and we would both be better off disarming. But each is afraid that if he throws away his weapons, the other one will not and he will be at a great disadvantage. So, when I saw that this... led to an armaments race, so to speak, I realized I was not the one to come up with a satisfactory theory... I keep on encouraging other people to work on it, though.
(Albers & Alexanderson, 1985, p. 26)

Question 6: What is the situation with the Soviet Union and the arms race that David Blackwell mentions? You may want to search the web in order to answer this question.

Question 7: What is the co-operative strategy for the arms race with the Soviet Union? What is the selfish strategy? Which strategy did the United States actually use?

Question 8: What are some other real-life situations that have similar elements? Explain in detail.